COMPLEX ANALYSIS FINAL EXAMINATION

Attempt all questions. Total Marks: 40. If you use a result proved in class then it is enough to just quote it. This exam is from 10 am to 12:30 pm (two and a half hours). You will also get some time before and after the examination for downloading and uploading respectively.

(1) Does there exist an analytic function f on the open ball $B(0;2)$ (with centre the origin and of radius 2) which satisfies $f(\frac{1}{n}) = \frac{1}{2^n}$ for all $n \in \mathbb{N}$? Justify your answer. (6 marks)

(2) For which positive integers n does there exist a primitive of the function $\frac{\sin(z)}{z^n}$ on $\mathbb{C} \setminus \{0\}$? Write down the Laurent series expansion centered at the origin, of any such primitive. (7 marks)

(3) Let $f(z) = (z - a_1)(z - a_2) \dots (z - a_{2n-1})(z - a_{2n})$ where $a_1, \dots, a_{2n}$ are distinct complex numbers. Let D be a domain in $\mathbb{C}$ satisfying a_{2k-1} and a_{2k} belong to a connected subset C_k of $\mathbb{C} \setminus D$ for each $1 \leq k \leq n$. Show that there exists a branch of the square root of f in D. (8 marks)

(4) Let $B = B(0;1)$ be the open unit ball $\{|z| < 1\}$, $\partial B = \{|z| = 1\}$ be the boundary of B, and $\bar{B} = \{|z| \leq 1\}$ be the closure of B. Let $f : \bar{B} \to \mathbb{C}$ be a non-constant continuous function which is analytic when restricted to B, and suppose that $|f(z)| = 1$ for all $z \in \partial B$. Show that f has a finite non-zero number of zeroes in B. Conclude that f is of the form

$$f = c \prod_{k=1}^{n} \left(\frac{z - a_k}{1 - \bar{a}_k z} \right)^{m_k}$$

where $a_1, \dots, a_n$ are distinct points of B, $m_1, \dots, m_n$ are positive integers, and c is a constant with $|c| = 1$. (Hint: Note that, if b is a complex number with $|b| < 1$, then $\frac{z-b}{1-bz} \leq 1$ if and only if $|z| \leq 1$, and equality holds if and only if $|z| = 1$) (9 marks)

(5) Let γ_n be the (closed) rectangular path $[n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni, n + \frac{1}{2} + ni]$, and consider the integral $\int_{\gamma_n} \pi(z + a)^{-2} \cot(\pi z) dz$ where a is not an integer. Show that

$$\lim_{n \to \infty} \int_{\gamma_n} \pi(z + a)^{-2} \cot(\pi z) dz = 0$$

Deduce that

$$\frac{\pi^2}{\sin^2(\pi a)} = \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2}$$

(Hint: For $z = x + iy$, use $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$, $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$, and show that $|\cot(\pi z)| \leq 2$ for any point z on the curve γ_n, when n is sufficiently large). (10 marks)